PLANE STRAIN IN THE ASYMMETRIC THEORY OF ELASTICITY

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The majority of works on the asymmetric theory of elasticity is devoted to the foundation of the theory and the derivation of fundamental relationships [1 to 5]. Equilibrium equations, Hooke's laws, and Saint Venant conditions extended to the couple-stress interaction case are obtained in these works. Neglecting volume force distributions and moments, the equilibrium equations are obtained as follows

$$\frac{\partial \sigma_{lk}}{\partial x_k} = 0, \qquad \frac{\partial \mu_{lk}}{\partial x_k} + \sigma_{nm} \in_{lmn} = 0 \tag{0.1}$$

Here σ_{lk} is the asymmetric stress tensor ($\sigma_{lk} \neq \sigma_{kl}$), μ_{lk} the micro-moments tensor, $\in_{lmn}$ the Levi-Civita tensor. The generalized Hooke's laws are taken as

$$\sigma_{lk} = \lambda E_{nn} \delta_{lk} + (\mu + \gamma) E_{kl} + (\mu - \gamma) E_{lk}, \qquad E_{lk} = \frac{\partial u_l}{\partial x_k} - \Omega_s \in_{skl} \tag{0.2}$$

$$\mu_{lk} = 2\eta r_{nn} \delta_{lk} + 2\tau r_{kl} + 2\theta r_{lk}, \qquad r_{lk} = \frac{\partial \Omega_l}{\partial x_k}$$

Here u is the displacement vector, Ω the rotation vector, which is an independent characteristic of the strain in the asymmetric theory of elasticity $\lambda, \mu, \gamma, \eta, \tau, \theta$ are material characteristics of the medium.

The generalized Saint Venant conditions become:

$$\frac{\partial r_{lk}}{\partial x_p} \in_{kpn} = 0, \qquad \frac{\partial E_{lk}}{\partial x_p} \in_{kpn} = r_{ss} \delta_{nl} - r_{nl} \tag{0.3}$$

Much less attention has been paid to the development of mathematical methods of solving the equations of the asymmetric theory of elasticity, which is an essential for elucidation of the specifics of the effects and the domain of applicability of this theory.

Taking account of the enormous mathematical difficulties originating in the solution of boundary value problems of the asymmetric theory of elasticity, we may proceed either by seeking correction terms to the solutions of problems of the ordinary theory of elasticity, or by developing exact methods, but for particular cases of body strain. The most important such strain case (as in the ordinary theory of elasticity) is the state of plane strain. The fundamental equations of plane strain have been obtained in [6]. Their solution is given there by the introduction of two fourth order functions. Their internal compatibility for compliance with the generalized Beltrami-Mitchell conditions produces an additional difficulty in finding the solution. A solution is obtained herein in terms of a fourth and second order function, needing no additional compatibility; a complex representation of this solution is given, and the question of separation of the boundary conditions is formulated.

1. Plane strain equations in the asymmetric theory of elasticity. We shall designate the strain plane if

$$u_x = u(x, y), \quad u_y = v(x, y), \quad u_z = 0, \quad \Omega_x = 0, \quad \Omega_y = 0, \quad \Omega_z = \Omega(x, y) \tag{1.1}$$

As is seen from (1.1), all the sections of an infinitely long prismatic body are deformed identically in plane strain, i.e., all particles are displaced in a plane perpendicular to the z-axis, and are rotated around the same axis. Taking account of (1.1) in (0.2), we obtain the generalized Hooke's laws for the plane strain case:

$$\sigma_{xx}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x},\qquad \sigma_{yy}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+2\gamma(w-\Omega),\qquad \sigma_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)-2\gamma(w-\Omega)$$

$$\sigma_{zz}=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right),\qquad \sigma_{xz}=\sigma_{yz}=\sigma_{zx}=\sigma_{zy}=0 \tag{1.2}$$

$$\mu_{xx}=\mu_{xy}=\mu_{yx}=\mu_{yy}=\mu_{zz}=0$$

$$\mu_{xz}=2\tau\frac{\partial\Omega}{\partial x},\qquad \mu_{yz}=2\tau\frac{\partial\Omega}{\partial y},\qquad \mu_{zx}=2\theta\frac{\partial\Omega}{zx},\qquad \mu_{zy}=2\theta\frac{\partial\Omega}{\partial y}$$

Here w is the rotation of a portion of the medium as a whole. The equilibrium Eqs. (0.1) simplify substantially if the relationships (1.2) are taken into account

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0,\qquad \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0,\qquad \frac{\partial\mu_{zx}}{\partial x}+\frac{\partial\mu_{zy}}{\partial y}+\sigma_{yx}-\sigma_{xy}=0 \tag{1.3}$$

The remaining equilibrium equations are satisfied identically. As regards the generalized Beltrami-Mitchell conditions, only three are retained:

$$\frac{\partial\mu_{zx}}{\partial y}-\frac{\partial\mu_{zy}}{\partial x}=0 \tag{1.4}$$

$$-\frac{(\mu-\gamma)}{4\mu\gamma}\frac{\partial\sigma_{xy}}{\partial x}+\frac{(\mu+\gamma)}{4\mu\gamma}\frac{\partial\sigma_{yx}}{\partial x}-\frac{1}{2\mu}\frac{\partial\sigma_{xx}}{\partial y}+\frac{\lambda}{4\mu(\lambda+\mu)}\frac{\partial}{\partial y}(\sigma_{xx}+\sigma_{yy})=\frac{1}{2\theta}\mu_{zx}$$

$$-\frac{(\mu-\gamma)}{4\mu\gamma}\frac{\partial\sigma_{xy}}{\partial y}+\frac{(\mu+\gamma)}{4\mu\gamma}\frac{\partial\sigma_{yx}}{\partial y}+\frac{1}{2\mu}\frac{\partial\sigma_{yy}}{\partial x}-\frac{\lambda}{4\mu(\lambda+\mu)}\frac{\partial}{\partial x}(\sigma_{xx}+\sigma_{yy})=\frac{1}{2\theta}\mu_{zy}$$

If (1.3) is taken into account, then they easily reduce to

$$\Delta(\sigma_{xx}+\sigma_{yy})=0 \tag{1.5}$$

$$\mu_{zy}=K_2^2\frac{\partial}{\partial y}(\sigma_{xy}-\sigma_{yx})+\frac{(\varkappa+1)\theta}{4\mu}\frac{\partial}{\partial x}(\sigma_{xx}+\sigma_{yy}),\qquad K_2^2=-\frac{(\mu-\gamma)\theta}{2\mu\gamma}$$

$$\mu_{zx}=K_2^2\frac{\partial}{\partial x}(\sigma_{xy}-\sigma_{yx})-\frac{(\varkappa+1)\theta}{4\mu}\frac{\partial}{\partial y}(\sigma_{xx}+\sigma_{yy}),\qquad \varkappa=\frac{\lambda+3\mu}{\lambda+\mu}$$

Therefore, Eqs. (1.3) and (1.4) are the fundamental plane strain equations in the asymmetric theory of elasticity. Let us note that such quantities as $\sigma_{zz}=\sigma(\sigma_{xx}+\sigma_{yy})$ (σ is the Poisson coefficient), μ_{xz} and μ_{yz} are found after the problem has been solved, i.e., in order for the body strain to be planar, definite σ_{zz}, μ_{xz} and μ_{yz} must be applied(*).

2. Solution of the equations. Let us seek the solution of (1.3) and (1.4) in terms of some auxiliary functions U and F, which represent the stresses as follows:

$$\sigma_{xx}=\frac{\partial^2U}{\partial y^2}-\frac{\partial^2F}{\partial x\,\partial y},\qquad \sigma_{yy}=\frac{\partial^2U}{\partial x^2}+\frac{\partial^2F}{\partial x\,\partial y}$$

$$\sigma_{xy}=-\frac{\partial^2U}{\partial x\,\partial y}+\frac{\partial^2F}{\partial x^2},\qquad \sigma_{yx}=-\frac{\partial^2U}{\partial x\,\partial y}-\frac{\partial^2F}{\partial y^2} \tag{2.1}$$

It is easy to see that the first two equilibrium equations are satisfied identically by virtue of (2.1). The last two Eqs. of the system (1.5) will be satisfied identically if they are considered as representations of μ_{zx} and μ_{zy} in terms of the functions U and F according to (2.1)

$$\mu_{zx}=\frac{\partial}{\partial x}(K_2^2\Delta F)-\frac{\nu}{2}\frac{\partial}{\partial y}\Delta U \tag{2.2}$$

$$\mu_{zy}=\frac{\partial}{\partial y}(K_2^2\Delta F)+\frac{\nu}{2}\frac{\partial}{\partial x}\Delta U,\qquad \nu=\frac{(\lambda+2\mu)\theta}{\mu(\lambda+\mu)}$$

*) If necessary to take account of volume forces and moments, it must be kept in mind that in plane strain they must have the form

$$X=X(x,y),\quad Y=Y(x,y),\quad Z=0,\quad m_x=0,\quad m_y=0,\quad m_z=m(x,y)$$

The first Eq. of (1.5) asserts that the quantity U is a biharmonic function

$$\Delta\Delta U = 0 \tag{2.3}$$

and the last relationship of the system (1.3) leads to the Eq.

$$\Delta (K_2^2 \Delta F - F) = 0 \tag{2.4}$$

which may be replaced by an equation of Helmholtz type (*)

$$K_2^2 \Delta F - F = 0 \tag{2.5}$$

Taking account of (2.5), as well as of the fact that $\frac{1}{2}\nu\Delta U$ is a harmonic function, and therefore, connected with its conjugate Q by means of the relationships

$$\frac{\partial Q}{\partial x} = -\frac{\nu}{2}\frac{\partial}{\partial y}\Delta U, \qquad \frac{\partial Q}{\partial y} = \frac{\nu}{2}\frac{\partial}{\partial x}\Delta U$$

(2.2) may be written as

$$\mu_{zx} = \frac{\partial}{\partial x}(F + Q), \qquad \mu_{zy} = \frac{\partial}{\partial y}(F + Q) \tag{2.6}$$

Therefore, under conditions (2.3) and (2.5), the Expressions (2.1) and (2.6) yield the general solution of Eqs. (1.3) and (1.5). Incidentally,the meaning of the function F, as a quantity proportional to the asymmetric part of the stress tensor

$$F = K_2^2 (\sigma_{xy} - \sigma_{yx})$$

follows from the expressions for σ_{xy} and σ_{yx} taking account of (2.5).

Expressions (2.1) and (2.6) may be written in complex form if the Goursat formula for the biharmonic function is utilized [7]:

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \chi(z)$$

Here $\varphi(z)$ and $\chi(z)$ are arbitrary analytic functions. Compactly written, (2.1) and (2.6) are

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \overline{\Phi}(z)] \qquad (\Phi(z) = \varphi'(z))$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] + 2\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y} + i\frac{\partial F}{\partial x}\right)$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{yx} = 2[\bar{z}\Phi'(z) + \Psi(z)] - 2i\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y} + i\frac{\partial F}{\partial x}\right)$$

$$\mu_{zy} - i\mu_{zx} = 2\nu\overline{\Phi'(z)} + \frac{\partial F}{\partial y} - i\frac{\partial F}{\partial x} \qquad (\Psi(z) = \chi''(z) = \psi'(z)) \tag{2.7}$$

It follows from the first two relationships of (2.7) that

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \overline{\Phi(z)} + z\overline{\Phi}'(z) + \overline{\Psi}(z) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y} - i\frac{\partial F}{\partial x}\right) \tag{2.8}$$

This expression may be integrated with respect to x, and expression obtained for the principal stress vector

$$i(X + iY) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \frac{\partial F}{\partial y} - i\frac{\partial F}{\partial x} \tag{2.9}$$

From (1.2) an expression is easily obtained for the displacement by neglecting the displacement of the body as a whole

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \frac{\partial F}{\partial y} + i\frac{\partial F}{\partial x} \tag{2.10}$$

and an expression for the angle of rotation Ω from the relationship for μ_{zx} without taking account of rotation of the body as a whole

$$2\theta\Omega = F - \nu i[\Phi(z) - \overline{\Phi(z)}] \tag{2.11}$$

The relationships (2.8) (or (2.9)), (2.10) and (2.11) and the last relationship in (2.7) yield expressions for the stress, the angle of rotation, the displacement and the micromoments in terms of two arbitrary analytic functions $\varphi(z)$ and $\chi(z)$ and a function F of Helmholtz type. It is easy to see that they are a generalization of the known Kolosov-Muskhelishvili relationships in the ordinary theory of elasticity.

*) The solution of (2.4) differs from the solution of (2.5) by a harmonic function which may be included in U.

3. Fundamental problems. Separation of boundary conditions for a half-plane. Problems of the ordinary theory of elasticity separate into three kinds:

1) Stresses are given on the contour (dynamic condition)
2) Displacements are given on the contour (kinematic condition),
3) Stresses are given on part of the contour, and displacements on the rest (mixed condition).

Such a classification for the problems of the ordinary theory of elasticity is natural since the state of stress is described by only the stress tensor, and the state of strain by the strain tensor (field of displacements). The state of stress in the asymmetric theory of elasticity is described by two dynamic quantities (the stress tensor σ_{lk} and the micromoments tensor μ_{lk}), and the state of strain, by two kinematic quantities (the displacement field $\mathbf{u}$ and the field of angles of rotation Ω). As has been shown in (2.7) to (2.11), these four quantities are expressed in terms of three arbitrary functions, two of which are harmonic ($\varphi\psi$) and one F is of Helmholtz type, hence, three out of the four arbitrary quantities

$$\sigma_{lk},\ \mu_{lk},\ \mathbf{u},\ \Omega$$

must be given on the contour to describe completely the state of stress or strain in the asymmetric theory of elasticity.

By analogy with the ordinary theory of elasticity, let us designate the first fundamental problem of the asymmetric theory of elasticity that for which the stresses and micromoments (dynamic conditions) are given on the contour, the second as that for which the displacements and angle of rotation (kinematic conditions) are given on the contour. Besides these fundamental problems, mixed problems (in the sense that partly dynamic, and partly kinematic quantities are given on the contour, as say: *a*) stresses and angle of rotation, *b*) displacements and micromoments) may be formulated in the asymmetric theory of elasticity, and we shall designate them as mixed problems of the second kind in the asymmetric theory of elasticity in contrast to the mixed problems of the first kind when one quantity is given on one part of the contour, and the other quantity is given on the other. Such problems in the asymmetric theory of elasticity are also greater in number than in the ordinary theory of elasticity, but they will not yet be considered.

The solution of the boundary value problems of the asymmetric theory of elasticity leads to greater mathematical difficulties as compared with the solution of boundary value problems of the ordinary theory of elasticity. The essential difficulty in boundary value problems of the asymmetric theory of elasticity is due to the "entanglement" of the boundary conditions for the harmonic functions $\varphi(z)$, $\psi(z)$ and the Helmholtz type function F.

However, for the half-plane the boundary conditions may be unraveled, i.e., the boundary value problem to determine $\varphi(z)$, $\psi(z)$ may be formulated separately, and to determine F separately. Let us show this by an example of the first fundamental problem of the asymmetric theory of elasticity, whose boundary value problems have the form

$$\begin{gathered} N(x) - iT(x) = \Phi(x) + \overline{\Phi(x)} + x\overline{\Phi'(x)} + \overline{\Psi}(x) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y} - i\,\frac{\partial F}{\partial x}\right)_{y=0} \\ M(x) = \nu\,[\Phi'(x) + \overline{\Phi'(x)}] + \left.\frac{\partial F}{\partial y}\right|_{y=0} \end{gathered} \qquad (3.1)$$

Here $N(x)$, $T(x)$, $M(x)$ are the boundary values of σ_{yy}, σ_{xy}, μ_{zy}, respectively, where it is considered that the body occupies the lower half-plane. The first relationship of (3.1) is the boundary value problem of the ordinary theory of elasticity to determine Φ and Ψ, if it is assumed that the boundary value of the last member of this equation is known. We find therefrom

$$\begin{gathered} \Phi(z) = \Phi_0(z) + \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y} - i\,\frac{\partial F}{\partial x}\right)\right]_{y=0}\frac{dx}{x-z} \\ \Phi_0(z) = -\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{N(x) - iT(x)}{x-z}\,dx \end{gathered} \qquad (3.2)$$

Substituting (3.2) into the second relationship of (3.1), we obtain

$$\left[\frac{\partial^3 F^*}{\partial x^3} + \frac{\partial^3 F}{\partial x^2 \partial y} - \frac{1}{\nu}\frac{\partial F}{\partial y}\right]_{y=0} = \left[\Phi_0'(x) + \overline{\Phi_0'(x)} - \frac{1}{\nu}\; M(x)\right]$$

$$F^*(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F(\zeta)}{\zeta - x}\, d\zeta \qquad (3.3)$$

Condition (3.3) is also a boundary condition to determine F. Knowledge of the latter reduces the solution of the first fundamental problem of the asymmetric theory of elasticity to the solution of the fundamental problem of the ordinary theory of elasticity, but with other boundary conditions, as is seen from the first relationship of (3.1). Analogously, the boundary conditions for the remaining boundary value problems of the asymmetric theory of elasticity for a half-plane may be unraveled. Let us present the form of the boundary conditions to find F when on the contour are given:

the displacements and the angle of rotation $u(x)$, $v(x)$, $\Omega(x)$

$$\left[\frac{\partial^2 F^*}{\partial x\, \partial y} - \frac{\partial^2 F}{\partial x^2} + \frac{\varkappa}{\nu} F\right]_{y=0} = \varkappa\left[\frac{2\theta}{\nu}\,\Omega(x) - i(\overline{\varphi}_0' - \varphi_0')\right] \qquad (3.4)$$

the stresses and the angle of rotation $N(x)$, $T(x)$, $\Omega(x)$

$$\left[\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F^*}{\partial x\, \partial y} + \frac{1}{\nu} F\right]_{y=0} = \left[i\left(\Phi_0(x) - \overline{\Phi_0(x)}\right) + \frac{2\theta}{\nu}\,\Omega(x)\right] \qquad (3.5)$$

the displacements and the micromoments $u(x)$, $v(x)$, $M(x)$

$$\left[\frac{\partial^3 F}{\partial x^2\, \partial y} + \frac{\partial^3 F^*}{\partial x^3} + \frac{\varkappa}{\nu}\frac{\partial F}{\partial y}\right]_{y=0} = \varkappa\left[\frac{1}{\nu}\, M(x) - (\varphi_0'' + \overline{\varphi}_0'')\right] \qquad (3.6)$$

Here Φ_0 and φ_0 denotes throughout the solution of the corresponding problems of the ordinary theory of elasticity.

The obtained boundary conditions (3.3) to (3.6) for F differ essentially from the known boundary value problems for Helmholtz type equations. Finding F with the boundary conditions (3.3) to (3.6) is a specific difficulty of the solution of problems of the asymmetric theory of elasticity and requires special consideration.

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